

Hydromagnetic stability of parallel flow of an ideal heterogeneous fluid

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The magnetohydrodynamic and Boussinesq approximations are used on a fully ionized ideal fluid with a longitudinal magnetic field. The flow is in a direction normal to the gravity vector and all variations in velocity, density and magnetic field. The stability characteristics, mainly for normal-mode perturbations, are investigated. Two simple problems with discontinuous profiles are solved analytically. For a double shear layer, an appropriate range of magnetic field values destabilizes the flow. A long wave theory is presented and applied to several problems, some of which are destabilized by an appropriate magnetic field. Finally, the solution for continuous profiles is presented and shown to decay algebraically in time for any stable stratification.

1. Introduction

There exists a rather extensive body of literature on the hydrodynamic stability of ideal parallel flows. A comprehensive review of work done up to 1966 is given by Drazin & Howard (1966). Only recently, however, has attention been focused on the effect that a magnetic field would have on the stability of parallel flow of an ionized fluid. The importance of such studies is indicated both by their intrinsic value and by their applicability to various astro- and geophysical phenomena such as flows within the sun's outer layers, the solar wind and the earth's magnetosphere (see e.g. Boller & Stolov 1970).

This theory would apply mostly to gaseous atmospheres. A scaling analysis is performed to determine the conditions under which the equations for a Boussinesq liquid can be applied to a gas. This also serves to simplify the analytical work.

Diffusive effects are neglected here because of the scale of the phenomena considered. We investigate the stability properties of an ideal magnetic fluid subjected to small perturbations. Nonlinear effects are not considered here.

Michael (1955) wrote the first paper on the magnetohydrodynamic stability of parallel flow, solving the magnetic version of the Kelvin–Helmholtz problem. The magnetic field had a stabilizing effect such that when the magnetic energy exceeded the kinetic energy the fluid was rendered completely stable. Subsequent investigators have shown that an appropriate magnetic field can also be

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destabilizing. Stern (1963) found that Couette flow can be rendered unstable by an appropriate piecewise-linear magnetic field. Kent (1966) showed how a variety of flows without inflexion points in the velocity profile, such as Poiseuille flow, would be destabilized by the addition of an appropriate magnetic field.

Kent (1968) investigated in some detail the general stability characteristics of a homogeneous magnetic fluid. By using the technique of Rosenbluth & Simon (1964), which incidentally can be used to solve Stern's problem, Kent found an example for which even a constant magnetic field caused destabilization. The velocity profile used had an inflexion point but did not satisfy Fjortoft's necessary condition for instability of a non-ionized fluid. One important but negative conclusion from this paper (Kent 1968) is that the theory of analytic marginally stable solutions, so useful in ordinary hydrodynamic stability of parallel flow, is restricted to a highly limited and artificial class of flows in the magnetic case.

In the papers by Stern and Kent, the destabilization found to be caused by the magnetic field was revealed by asymptotic techniques at long wavelength. Actual growth speeds were not found. In this paper a double shear-layer problem is solved analytically and is shown to be more unstable than the non-magnetic problem at various values of the parameters. Analysis of this problem sheds light on the nature of the destabilizing effect, although the interpretation cannot be extended with complete freedom to problems with continuous velocity distributions because the energy transfer terms differ in the two cases.

The generally stabilizing influence which the magnetic field exerts on the fluid is due to a tension which usually acts as a restoring force on disturbances. Nevertheless, in the non-magnetic problem there is an inherent inefficiency in the instability mechanism for most velocity profiles. The perturbations do not make use of all the kinetic energy available in the basic flow, as is clearly revealed for disturbances of short wavelength in the double shear-layer model. Disturbances at one interface are virtually independent of those on the other. When a magnetic field is included, the entire fluid is 'tied together' and this has the effect of making available the entire kinetic energy of the flow. For Stern's problem it is the magnetic Reynolds stresses which contribute to the instability, but such terms do not appear in discontinuous models.

The stability characteristics of a magnetic fluid are extended throughout this paper by the incorporation of variable-density effects. The occasional destabilizing influence of the magnetic field also extends to the heterogeneous case. Larger values of the Richardson number at marginal stability are found for some velocity profiles for appropriate magnetic field configurations.

Many of the stability characteristics are revealed by an extension to the magnetic problem of the long-wave theory of Drazin & Howard (1961, 1962). An approximate eigenvalue relation is derived and yields the exact formulae for the wave speeds of the three-layer models to first order in the wavenumber.

Finally, the continuum solution is given for flows in which the kinetic energy exceeds the magnetic energy. The fluid is stable for stable stratification and the perturbation shrinks algebraically with time. For zero stratification the disturbances are neutral.

2. The basic problem

We investigate the stability properties of a fully ionized, heterogeneous, non-dissipative shearing fluid. Cartesian co-ordinates (x, y, z) are employed throughout the paper. Using a velocity scale V , the basic dimensionless velocity is given by U . The dimensionless magnetic field M is given by $B = (4\pi\rho_0)^{\frac{1}{2}} VM$, where ρ_0 is the density of the basic state. The parameters describing the basic state are functions of z only while the flow and magnetic field direction is along the x axis. Gravity g is assumed to be constant and is taken in the z direction.

The governing equation for the stability of normal-mode perturbations for the homogeneous case ($\rho_0 = \text{constant}$) have been derived in several forms by Kent (1968) among others. The generalization of these equations to include the effects of stratification under the Boussinesq approximation is derived by Gedzelman (1970) and appears here as three equations, namely,

$$D \left[\frac{X}{(U-c)^2} DW \right] - \left\{ k^2 \left[\frac{X}{(U-c)^2} \right] + \frac{1}{U-c} D \left[DU \frac{X}{(U-c)^2} \right] + \bar{R}_i \right\} W = 0, \quad (2.1)$$

$$D[XDF] - [k^2X + \bar{R}_i]F = 0, \quad (2.2)$$

$$D^2Y - \left[k^2 + \frac{2XD^2X - (DX)^2}{4X^2} + \frac{\bar{R}_i}{X^{\frac{1}{2}}} \right] Y = 0, \quad (2.3)$$

where D is the z derivative, W the perturbation vertical velocity, $F \equiv W/(U-c)$, the perturbation amplitude, $c = c_r + ic_i$ is the complex wave speed, k is the wave-number, $Y = X^{\frac{1}{2}}F$, $X = (U-c)^2 - M^2$, and $R_i = gD\rho_0/\rho_0(DU)^2 \equiv \bar{R}_i(DU)^2$ is the Richardson number. The normal-mode solution assumed has a functional form given by

$$F(x, z, t) = \tilde{F}(z) e^{ik(k-ct)}.$$

When the fluid is contained by walls at $z = z_1, z_2$ the perturbation vertical velocity vanishes there and the boundary conditions are simply

$$W(z) = 0 \quad \text{at} \quad z = z_1, z_2.$$

Variations in the y direction have not been included because an extension of Squire's theorem is valid (see Gedzelman 1970). Furthermore, as with the non-magnetic problem the existence of a particular complex wave speed implies the existence of its complex conjugate. Thus, instability corresponds to any solution with a non-real wave speed and stability or, strictly speaking neutrality corresponds to solutions with real wave speeds.

Many of the integral theorems used on the non-magnetic problem lose their usefulness when extended to the magnetic problem. Only the semicircle theorem (see e.g. Gilman 1967) remains of much value. This places a greater restriction on both the maximum possible growth rate and the range of phase speeds for unstable waves but does not imply that the magnetic field is always stabilizing.

In the non-magnetic homogeneous problem we know that, for marginal stability, $U = c$ at the inflexion point. Inspection of the governing equation shows that, because of this the coefficients are not singular. Analogously, one would

expect to find a marginally unstable solution when the coefficients of (2.3) are not singular. The quantity $2XD^2X - (DX)^2$ must therefore have zeros of the same order as X^2 . Using a Taylor series expansion around z_0 , where $X(z_0) = 0$, we have

$$2[X_0 + \eta DX_0 + \dots][D^2X_0 + \eta D^3X_0 + \dots] - [DX_0 + \eta D^2X_0 + \dots]^2,$$

where $\eta = z - z_0$ and the subscript indicates evaluation at z_0 . To zeroth order we have $2X_0D^2X_0 - (DX_0)^2 = 0$ and since $X_0 = 0$, so does DX_0 . Noting that $X_0 = DX_0 = 0$ we then have to higher order in η

$$\eta^3[\frac{1}{3}D^3X_0D^2X_0] = 0, \quad \eta^4[\frac{1}{6}D^2X_0D^4X_0 + \frac{1}{2}(D^3X_0)^2] \neq 0.$$

If $D^2X_0 = 0$ then the coefficient of η^4 is also non-zero and $D^3X_0 = 0$. This implies that

$$U_0 = c, \quad M_0 = 0, \quad D^2U_0DM_0 - D^2M_0DU_0 = 0,$$

are conditions of marginal stability as Kent (1968) has already shown through more rigorous argument. Thus the theory of analytic marginally stable solutions is restricted to a highly artificial and limited class of profiles for the magnetic problem. Generally, the governing equation has solutions with logarithmic singularities and it is perhaps this feature which further complicates an already difficult problem. Logarithmic terms are seen to appear in the problems attacked in §4.

Equations (2.1)–(2.3) were derived with a liquid in mind. It is appropriate to perform a scaling analysis to see under what conditions these equations will describe the behaviour of a gas.

When the fluid under consideration is an ideal gas the equation of state becomes

$$P = \rho RT, \tag{2.4}$$

where P is the pressure, R is the ideal-gas constant and T is the absolute temperature. The thermal equation now describes the conservation of potential temperature;

$$d\theta/dT = 0, \tag{2.5}$$

where θ is defined by Poisson's equation

$$\frac{T}{\theta} = \left(\frac{P}{P_{\text{ref}}}\right)^{R/c_p}. \tag{2.6}$$

Once again the equations are non-dimensionalized in order to determine under what conditions (2.2) is valid for a gas. The velocity scale is again given by V . We take ϵ to be the amplitude of the velocity perturbation. An average magnitude for a variable is indicated by an overbar. We write

$$\bar{P} = \bar{\rho}R\bar{T}, \quad P_a = \bar{P}P_s, \quad \rho_a = \bar{\rho}\rho_s,$$

where subscript a indicates undisturbed state.

The variables are scaled in the following manner:

$$\begin{aligned} P &= \bar{P}P_s(1 + \tilde{p}p_1 + \tilde{p}^2p_2 + \dots), \\ \rho &= \bar{\rho}\rho_s(1 + \tilde{\rho}\rho_1 + \tilde{\rho}^2\rho_2 + \dots), \\ \theta &= \bar{\theta}\theta_s(1 + \tilde{\theta}\theta_1 + \tilde{\theta}^2\theta_2 + \dots), \end{aligned}$$

where the tilde indicates the perturbation amplitude. The magnetic field B is scaled to be

$$B = (4\pi\bar{\rho})^{\frac{1}{2}} V(B_0 + \epsilon b_1 + \epsilon^2 b_2 + \dots).$$

The precise value of \bar{P} is determined by balancing the pressure term with the inertial term in the horizontal momentum equation, and thus

$$\bar{P} = \epsilon V^2 \bar{\rho} / \bar{P}.$$

From Poisson's equation we find that

$$\tilde{\rho} = \tilde{\theta} = (c_v/c_p) \tilde{P}.$$

Dropping the subscript 1, the scaled continuity equation appears as

$$\tilde{\rho} \left[\frac{\partial \rho}{\partial t} + U \frac{\partial \rho}{\partial x} \right] + \frac{W}{\rho_s} \frac{\partial \rho_s}{\partial z} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}. \quad (2.7)$$

In order that the continuity equation should take the form

$$\partial u / \partial x + \partial w / \partial z = 0,$$

we must demand that

$$\frac{1}{\rho_s} \frac{\partial \rho_s}{\partial z} \ll 1; \quad \frac{c_p}{c_v} V^2 \frac{\bar{\rho}}{\bar{P}} \ll 1. \quad (2.8)$$

Defining a new magnetic field such that

$$M = B_0 / (\rho_s)^{\frac{1}{2}}, \quad h = b / (\rho_s)^{\frac{1}{2}},$$

and a new pressure

$$\Pi = P P_s / \rho_s + M h_x,$$

and stating that

$$d\theta/dz \equiv \beta(z),$$

we find that the remaining equations reduce to those appropriate for a Bousinesq liquid as long as inequalities (2.8) are satisfied. The first inequality requires that the percentage change in the density be small and the second requires that the basic velocity be much smaller than the equivalent shallow-water wave speed, i.e. that the Froude number be much smaller than unity;

$$\frac{c_p}{c_v} V^2 \frac{1}{RT} \simeq \frac{c_p}{c_v} V^2 \frac{1}{gH} \ll 1.$$

When the inequalities (2.8) are obeyed, (2.2) is a valid form for the governing equation of an ideal gas.

3. Problems

The magnetic analogue of the Kelvin-Helmholtz problem was solved by Michael (1955). The magnetic field exerts a stabilizing influence owing to the 'tension' which it imparts to the fluid. Models with continuous profiles of velocity, density and magnetic field behave in a more complicated manner than does this simple two-layer model. In this section two three-layer problems are solved. These exhibit several characteristics of flows with continuous profiles but retain the simplicity of solution characteristics of the Kelvin-Helmholtz problem.

In each layer of constant U , ρ and M the governing equation (2.1) reduces to

$$k(U-c) \left[1 - \frac{M^2}{(U-c)^2} \right] \{ D^2 W - k^2 W \} = 0. \quad (3.1)$$

Boundary conditions at $z = \pm \infty$ require the perturbation vertical velocity to remain finite. Interfacial boundary conditions call for the continuity of normal velocity and continuity of pressure and are given respectively as

$$\Delta_s [W/(U-c)] = 0, \quad (3.2)$$

$$0 = \Delta_s \left[-(U-c) DW + M^2 D \left(\frac{W}{U-c} \right) + \frac{g\rho}{\rho_0} \left(\frac{W}{U-c} \right) \right], \quad (3.3)$$

where Δ_s indicates the jump in the bracketed quantity across the interface.

The three-layer jet

This problem has been considered by Axford (1960) without a magnetic field in the inner region. It is described by

$$U = \begin{cases} 0, & |z| > 1, \\ 1, & |z| < 1; \end{cases} \quad M = \begin{cases} M_1, & |z| > 1, \\ M_0, & |z| < 1; \end{cases} \quad \rho = \begin{cases} \rho_0 - \Delta\rho, & z > 1, \\ \rho_0, & |z| < 1, \\ \rho_0 + \Delta\rho, & z < -1. \end{cases}$$

The resulting wave speeds are given by

$$c_1 = \frac{1}{2}(2-a^2) \pm \frac{1}{2}[(2-a^2)^2 - (2-a^2)p + 2a^2q]^{\frac{1}{2}}, \quad (3.4)$$

$$c_2 = a^2 \pm \frac{1}{2}[a^4 - 2a^2p + 2(2-a^2)q]^{\frac{1}{2}}, \quad (3.5)$$

and represent the varicose and sinuous waves respectively. Here

$$p \equiv 1 - M_0^2, \quad q \equiv M_1^2 + G/k, \quad G \equiv (\Delta\rho/\rho_0)g, \quad a^2 \equiv 1 - e^{-2k}.$$

In the long wavelength limit ($a^2 \rightarrow 0$) of the homogeneous non-magnetic problem both these waves are marginally stable (unstable for $k > 0$). For $k = 0$ the varicose wave travels with the maximum velocity of the current and the sinuous wave with the minimum.

For the homogeneous case at $k = 0$ the sinuous waves travel with speed $\pm M_1$ and are completely stable. From (3.5) we find that any $M_1^2 \geq \frac{1}{2}$ or any $M_0^2 \geq 1.0$ rules out instability and, as with the varicose wave, the magnetic field serves only as a stabilizing agent.

For $M_0^2 > M_1^2$ the sinuous wave dominates while for $M_1^2 > M_0^2$, the varicose wave dominates. Not much is gained by investigating the stratified problem which also makes clear the stabilizing influence of the magnetic field.

At this point it is appropriate to present an extension of a heuristic formula developed by Baekus (as presented by Drazin & Howard (1966)) to the magnetic problem for finding the wave speed for small k of the sinuous disturbances in an unbounded fluid. The logic behind the argument is that, for sufficiently long wavelength disturbances of the sinuous type, the jet can be treated essentially as a string. Disturbances die out with a scale height of $1/k \gg L$, where L is the width of the jet. The height of the disturbance is assumed to be given approximately by

$$\eta = \eta_0 e^{-k|z|} e^{ik(x-ct)}. \quad (3.6)$$

A force balance equation may be written in the z direction for half a wavelength as

$$\int \rho \frac{\partial^2 \eta}{\partial t^2} dz = \int \rho \frac{U^2}{r} dz - g \int \Delta \rho \eta dz + \int \rho M \frac{\partial h_z}{\partial x} dz, \tag{3.7}$$

where the inertial term is balanced, from left to right, by centrifugal, buoyancy and magnetic forces. r is the radius of curvature. If (3.6) is substituted into (3.7) the inertial term becomes

$$\int \rho \frac{\partial^2 \eta}{\partial t^2} dz = -2kc^2 \bar{\rho} \eta_0.$$

The centrifugal term may be rewritten as

$$\int \rho \frac{U^2}{r} dz \cong \int \rho U^2 \frac{\partial^2 \eta}{\partial x^2} dz \cong k^2 \bar{\rho} \eta_0 \int U^2 dz,$$

since the major contribution comes from the vicinity of the jet (where $U \equiv 0$). The buoyancy force becomes simply

$$-g \int \Delta \rho \eta dz \cong g(\rho_\infty - \rho_{-\infty}) \eta_0.$$

Using the fact that the magnetic field lines remain parallel to the flow,

$$h_z = M \partial \eta / \partial x$$

and writing $M \equiv M_B + M_v$ (a background plus a variable part), we get

$$\int \rho M \frac{\partial h_z}{\partial x} dz \cong -\bar{\rho} k^2 \eta_0 \int M_v (2M_B + M_v) dz - 2M_B^2 k \bar{\rho} \eta_0,$$

and (3.7) becomes

$$c^2 = -\frac{k}{2} \int_{-\infty}^{\infty} [U^2 - M_v (2M_B + M_v)] dz + M_B^2 + \frac{g(\rho_{-\infty} - \rho_\infty)}{2\bar{\rho}k}. \tag{3.8}$$

Applying this to the three-layer jet model, we obtain

$$c^2 = -k[1 - M_0^2 + M_1^2] + G/2k + M_1^2.$$

To first order for small k , the discriminant of (3.5) gives exactly the same result. We shall have further cause to refer to this argument when considering the general expansion technique for long wavelengths. Let us note here that any basic magnetic field is sufficient to stabilize long wave, sinuous disturbances in such a model simply because there is not enough kinetic energy of the basic flow available to be converted into the magnetic energy of the disturbances.

Double shear layer

We now consider the antisymmetric double shear layer given by

$$U = \begin{cases} 1, & z > 1, \\ 0, & |z| < 1, \\ -1, & z < -1; \end{cases} \quad M = \begin{cases} M_0, & |z| < 1, \\ M_1, & |z| > 1; \end{cases} \quad \rho = \begin{cases} \rho_0 - \Delta\rho, & z > 1, \\ \rho_0, & |z| < 1, \\ \rho_0 + \Delta\rho, & z < -1. \end{cases}$$

Because the flow indicates no preferred direction, it seems plausible that instability will set in as a wave with $c_r = 0$ so that the principle of 'exchange of stabilities' would then be valid. Although this is often the case for antisymmetric flows, it need not be so. Howard (1963) showed that the curve $c = 0$ in the G, k plane does not define the stability boundary for the antisymmetric double shear layer without a magnetic field and that instability sets in as two waves travelling with equal but opposite velocities. While there are unstable solutions with $c_r = 0$, these lie embedded entirely within the unstable region.

When an aligned magnetic field is superposed on this pattern several new interesting features arise. The magnetic field actually destabilizes the problem in many instances. This example therefore proves to yield the first complete analytical solution to an ideal magnetic problem of parallel flow in which the magnetic field destabilizes the fluid. There are even ranges of the parameters for which this destabilization occurs when the magnetic field is constant. We now proceed to present an analysis of the problem.

After the kinematic boundary conditions have been satisfied the solution for W is given by

$$W = \begin{cases} (A + B e^{2k}) (1 - c) e^{-k(z-1)}, & z \geq 1, \\ Ac e^{2k} e^{-k(z+1)} - Bc e^{2k} e^{k(z-1)}, & |z| \leq 1, \\ -(A e^{2k} + B) (1 + c) e^{k(z+1)}, & z \leq -1. \end{cases} \quad (3.9)$$

The dynamical boundary conditions at $z = \pm 1$ respectively are given by

$$[(1 - c)^2 - c^2 - M_1^2 + M_0^2 - G/k] A e^{-2k} + [(1 - c)^2 + c^2 - M_1^2 - M_0^2 - G/k] B = 0, \quad (3.10a)$$

$$\text{and} \quad [c^2 + (1 + c)^2 - M_0^2 - M_1^2 - G/k] A + [(c + 1)^2 - c^2 + M_0^2 - M_1^2 - G/k] B e^{-2k} = 0. \quad (3.10b)$$

$$\text{We define} \quad M \equiv 1 - M_1^2 - G/k, \quad a^2 \equiv 1 - e^{-4k}.$$

The solution for the eigenvalue c is obtained by substituting (3.10) into (3.9) and we have

$$c^2 = -\frac{1}{2}(m - M_0^2 - a^2) \pm \frac{1}{2}(a^4 - 2a^2(m - M_0^2) + (1 - a^2)[M + M_0^2]^2)^{\frac{1}{2}}. \quad (3.11)$$

Instability can arise in any of three manners. Since (3.11) is of the form

$$c^2 = S \pm T^{\frac{1}{2}},$$

we see that c has an imaginary part if $S < 0$ or $T > S^2$ or $T < 0$ and thus the wave is unstable. An example of a stability diagram is presented in figure 1, where we use the case of $M_0^2 = 0.2$ and $M_1^2 = 0$. As long as we are within a region bounded by any one of the three marginal stability curves $T = 0$, $S = 0$ and $T = S^2$, there is an instability.

Before analysing the problem in complete generality, let us consider the homogeneous case with no background magnetic field ($G = 0$, $M_1^2 = 0$). Equation (3.11) then becomes

$$c^2 = \frac{1}{2}(a^2 + M_0^2 - 1) + \frac{1}{2}[a^4 - 2a^2(1 - M_0^2) + (1 - a^2)(1 + M_0^2)^2]^{\frac{1}{2}}. \quad (3.12)$$

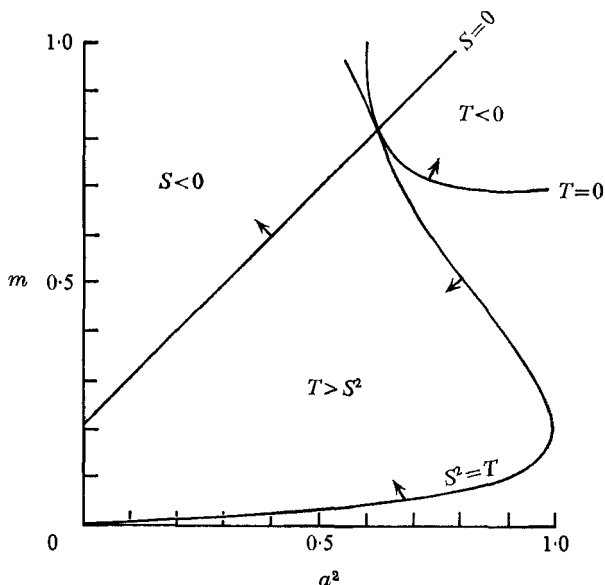


FIGURE 1. Schematic diagram for marginal stability curves of double shear layer. Arrows indicate the direction of instability.

Instability will arise from $S < 0$ whenever $a^2 < a_S^2 \equiv 1 - M_0^2$ and will be manifest with $c_r = 0$. For $M_0^2 > 1.0$ this mode disappears. Instability will occur through $T > S^2$ whenever

$$a^2 < a_{T,S}^2 \equiv 4M_0^2 / (1 + M_0^2)^2.$$

It is seen that as the magnetic field approaches zero this means of producing an instability vanishes. The third mode for producing an unstable wave occurs when $T < 0$. Here c^2 and hence c is complex so that $c_r \neq 0$. Since increasing M_0^2 serves only to increase T , we find that, when $M_0^2 > \frac{1}{2}$, $T > 0$ and instability will be manifest by stationary waves only ($c_r = 0$). Instability for $T < 0$ will occur for all

$$a^2 > a_T^2 \equiv \frac{1}{2} \{ 3 + M_0^4 - [(3 + M_0^4)^2 - 4(M_0^2 + 1)^2]^{\frac{1}{2}} \}.$$

For $0.295 < M_0^2 < 0.5$ we find that $0 < a_S^2 < a_{T,S}^2 < a_T^2 < 1$. Thus for an intermediate range of wavelengths $a_{T,S}^2 < a^2 < a_T^2$, the waves are stable. Instability for short wavelengths is governed by $T < 0$ and for long wavelengths by $T > S^2$.

Growth rate curves for various values of M_0^2 are given in figures 2(a) and (b). In the case $M_0^2 = 1.0$ greater instability for all $a^2 < 0.718$ ($k < 0.316$) is realized than for the non-magnetic problem. At $a^2 = 0$ we have

$$c^2 = \frac{1}{2}(M_0^2 - 1) \pm \frac{1}{2}(M_0^2 + 1).$$

The negative root results in instability no matter how large the magnetic field M_0^2 . Furthermore, for values of M_0^2 as great as 5.0 the magnetic problem has greater instability than the non-magnetic problem at sufficiently long wavelengths.

We may obtain some insight into the destabilizing influence. When $a^2 \equiv 1$, we find for $M_0^2 = 0$ that

$$c^2 = \frac{1}{2}i; \quad c = \frac{1}{2}(1 + i).$$

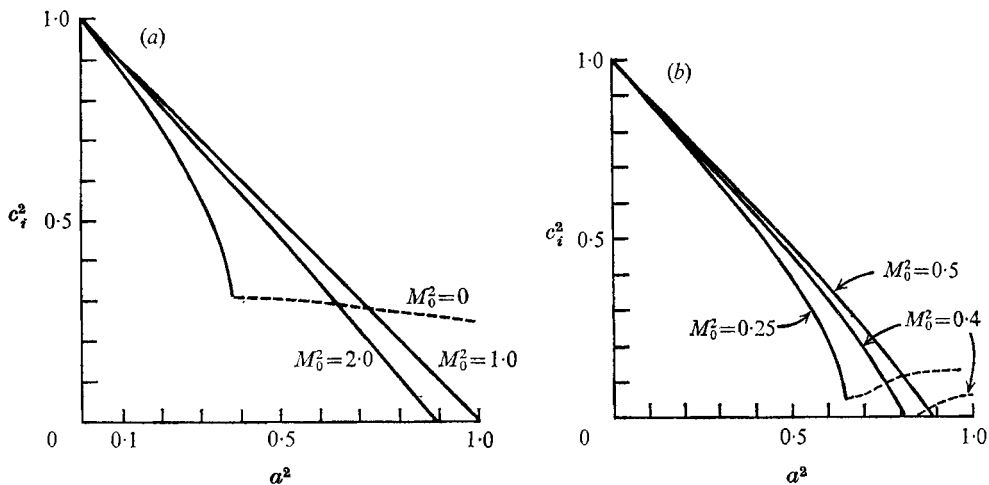


FIGURE 2. Growth rates for homogeneous double shear layer with $M_1 = 0$;
 (a) $M_0^2 = 0, 1, 2$. (b) $M_0^2 = 0.25, 0.4, 0.5$. —, $c_r = 0$; ---, $c_r \neq 0$.

Substituting this into (3.10b) yields $A \sim B e^{-2k}$. Therefore, by (3.17), any solution of magnitude $B e^{2k}$ at $z = 1$ has magnitude B at $z = -1$. Since k is large, the magnitude of any solution decreases rapidly from either interface. We have, as Howard (1963) mentioned, two essentially separate instabilities, one at each interface, travelling at the average velocity around that interface. For the magnetic problem the situation is different. Taking the wave speed for the case $M_0^2 = 1$ we have

$$c^2 = -e^{-4k}; \quad c = i e^{-2k}.$$

Upon substituting c into (3.10) we obtain $A \sim B$, so that the disturbance maintains its amplitude throughout the intermediate layer. Thus, there are two influences that the magnetic field exerts on a fluid. A magnetic field in a fluid of infinite conductivity adds a tension to the fluid so that it becomes more difficult to produce an instability. But the field also adds a cohesiveness to the fluid which it may not have possessed before. This cohesiveness may serve either to increase or produce instability by making available an extra energy source which the non-magnetic fluid could not have used. Thus the two semi-infinite layers are always tied together in the magnetic problem so that there is more kinetic energy of the basic flow available for transformation to perturbation energy than in the non-magnetic problem. It must be mentioned that this 'cohesiveness' is, in fact, the tension of the magnetic field lines, so that the two influences spring from the same source, but it is clear that the behaviour of this one force may manifest itself in a variety of ways.

We now generalize the discussion to include the effects of a background magnetic field M_1 and stratification G . For this problem $m < 1$. Consider first the case $M_0^2 = 0$. We find that $S^2 - T = a^2 m^2$, so that instability may arise by only two means ($S < 0, T < 0$). For $S > 0$ instability occurs only with $T < 0$, i.e. when

$$m > m_T = a^2 / (a + 1).$$

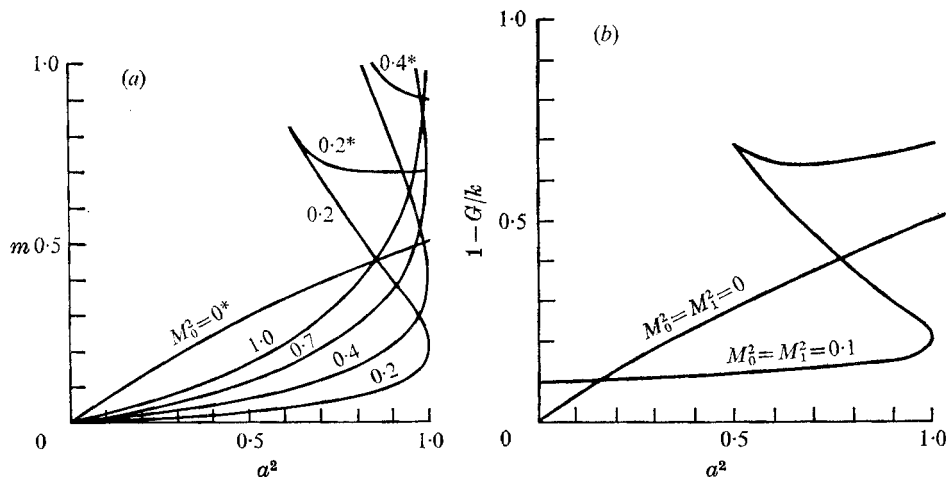


FIGURE 3. Marginal stability curves for (a) various M_0^2 values (numbers by curves), $M_1 = 0$; (b) $M_0^2 = M_1^2 = 0$; $M_0^2 = M_1^2 = 0.1$.

When we include the effects of M_0^2 we find that T may be greater than S^2 . Setting $S^2 = T$ we obtain

$$m_{T,S} = M_0^2[(2 - a^2) \pm 2(1 - a^2)^{\frac{1}{2}}]/a^2.$$

For all m with values between these two roots there is instability. At $M_0^2 = 0$ this mode is degenerate, so that no instability may occur in this way for the non-magnetic problem. Instability is guaranteed for $S < 0$ or for $m > m_S \equiv a^2 + M_0^2$. Finally, when $T < 0$ we have instability with $c_r = 0$ and this occurs when m takes values between the two roots

$$m_T = -[(1 - a^2)2M_0^2 + 2a^2 \pm 2(2a^2(1 - a^2)M_0^2(1 - M_0^2) + a^6)^{\frac{1}{2}}]/2(1 - a^2).$$

Beyond $M_0^2 = \frac{1}{2}$ this source of instability vanishes.

A comparison of the marginal stability curves for various M_0^2 values is given in figure 3(a). The apparently unreasonable result that, as M_0^2 decreases, the bottom line of the stability boundary departs further from the stability boundary for $M_0^2 = 0$ is resolved by noticing that the curves designated with the asterisk (* which corresponds to the mode $T < 0$) coincide with the $M_0^2 = 0$ curve in the limit of $M_0^2 \rightarrow 0$. We see that the presence of an appropriate magnetic field in the central layer may destabilize an otherwise stable configuration for the non-magnetic problem at any wavenumber and for any value of G/k up to 1.0.

The application of a constant magnetic field may also destabilize the double shear-layer problem for a considerable range of wave-numbers. Consider the situation depicted in figure 3(b). We see that the neutral curve for $M_0^2 = M_1^2 = 1$ lies above that for $M_0^2 = M_1^2 = 0.1$ for all $a^2 > 0.15$, so that instability occurs for greater stratification with a magnetic field for all reasonably short waves. As the magnetic field increases destabilization occurs for an increasingly restricted range of wavenumbers until at $M^2 \cong 0.3$, where the destabilizing effect vanishes.

4. Long wave theory

We now consider the stability characteristics of long wavelength disturbances in an unbounded fluid. The approach used here represents an extension of the treatments of Drazin & Howard (1961, 1962) to a magnetic fluid. There are two equivalent techniques which may be used to obtain an approximate equation for the complex wave speed. One is an integral equation approach and the other is a double series expansion given in terms of the wavenumber k and the stratification G .

The eigenvalue relation

At wavenumber zero, any finite stratification will stabilize the flow. It is necessary to use a double series expansion so that one can determine the critical stratification at small wavenumbers.

It is convenient to express our governing equation (2.3) as

$$D[XDF] - k^2XF + GD\lambda F = 0, \quad (4.1)$$

where we have defined

$$G \equiv \frac{\rho_{-\infty} - \rho_{\infty}}{\rho_{-\infty} + \rho_{\infty}} g, \quad D\lambda \equiv \frac{\bar{R}_i}{G}.$$

The subscripts $\pm\infty$ are used throughout this chapter to denote the z values at which the parameters are given. Equation (4.1) is valid as long as $G/g \ll 1$, i.e. fractional variations in density are small.

A straightforward double expansion scheme given in powers of k and G does not converge uniformly for an unbounded fluid. An alternative procedure is necessary. Under the conditions that $X, G \rightarrow \text{constant}$ as $|z| \rightarrow \infty$ and

$$\int_{-\infty}^{\infty} \left| \frac{2XD^2X - (DX)^2}{X^2} + \frac{GD\lambda}{X^{\frac{1}{2}}} \right| dz$$

converges, it is guaranteed that, as $|z| \rightarrow \infty$, equation (2.3) becomes

$$D^2Y - k^2Y = 0,$$

so that asymptotically Y behaves like $e^{-k|z|}$. Because X approaches a constant, F exhibits the same asymptotic behaviour as Y and we thus find it convenient to write

$$F_+ = e^{-kz}\xi(z), \quad F_- = e^{kz}\phi(z). \quad (4.2)$$

These two solutions must match at $z = 0$, so that the internal boundary condition is

$$(F_+DF_- - F_-DF_+)|_{z=0} = 0. \quad (4.3)$$

Substitution of (4.2) into (4.3) yields

$$\xi(0)D\phi(0) - \phi(0)D\xi(0) - 2k\xi(0)\phi(0) = 0. \quad (4.4)$$

The double series expansions for $\xi(z)$ and $\phi(z)$, are written as

$$\xi(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \xi_{i,j} k^i G^j, \quad \phi(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \phi_{i,j} k^i G^j. \quad (4.5)$$

When we substitute (4.2) and (4.5) into (4.1) and equate powers of k and G we obtain the following recursion relations for $\xi_{i,j}$ and $\phi_{i,j}$:

$$D[XD\phi_{i,j}] = -D[X\phi_{i-1,j}] - XD\phi_{i-1,j} + D\lambda\phi_{i,j-1}, \tag{4.6}$$

where $\phi_{i,j} = 0$ for $i < 0$ or $j < 0$, and

$$D[XD\xi_{i,j}] = D[X\xi_{i-1,j}] + XD\xi_{i-1,j} + D\lambda\xi_{i,j-1}, \tag{4.7}$$

where $\xi_{i,j} = 0$ for $i < 0$ or $j < 0$.

After some manipulation of the second-order terms the eigenvalue formula becomes

$$k[X_\infty + X_{-\infty}] - 2G + \int_{-\infty}^{\infty} \frac{dz}{X} [k(X - X_\infty) + G(1 - \lambda)] \times [k(X - X_{-\infty}) + G(1 + \lambda)] + \dots = 0. \tag{4.8}$$

As long as the growth speed is non-zero this series is convergent. The proof is somewhat longer than that given by Drazin & Howard (1962) but follows essentially the same line of reasoning. For the details the reader is referred to Gedzelman (1970).

An expression equivalent to (4.8) can be obtained for jet profiles by using an integral equation approach. Once again this represents a straightforward extension of the argument of Drazin & Howard (1962) to the case of a stratified, magnetic fluid. The eigenvalue relation is given by setting the Fredholm determinant equal to zero, i.e.

$$\mathcal{D} = 1 + \frac{k}{2X_\infty} \int_{-\infty}^{\infty} K(z, z) dz + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{k^n}{2^n X_\infty^n} \int dz_1 \dots \int dz_n \det |K(z_i, z_j)| = 0,$$

where

$$K(z_i, z_j) = \left\{ \frac{1}{X(z_j)} [X^2(z_j) - X_\infty^2] - \frac{G}{k^2} D\lambda \right\} \exp \left[-kX_\infty \operatorname{sgn}(z_i - z_j) \int \frac{dz}{X} \right].$$

The first two terms for the eigenvalue relation are given by

$$1 + \frac{k}{2c} \int_{-\infty}^{\infty} \left[\frac{(X - X_\infty)^2}{X} - \frac{G}{k^2} D\lambda \right] dz + \dots = 0.$$

Examples

At this point it is of interest to see if (4.8) may be applied with success to actual examples for which complete solutions do exist. It is unfortunate that for magnetic fluids the only extant solutions are for problems with discontinuous profiles. Nevertheless, close agreement between the results of (4.12) and the actual solutions should be quite encouraging since (4.12) is based on a series expansion better suited to describing continuous profiles.

Recalling from §3 that velocities have been normalized, (4.8) yields to first order

$$k[2c^2 - M_\infty^2 - M_{-\infty}^2 + 2] - 2G = 0,$$

which is exact for the Kelvin-Helmholtz problem. For the double shear-layer model equation (4.8), when solved for c^2 , yields

$$c^2 = -\frac{1}{2}(1 - M_0^2 - M_1^2 - G/k - 4k) \pm \frac{1}{2}[(1 - M_0^2 - M_1^2 - G/k - 4k) - 4k(1 - M_1^2 - G/k + M_0^2)^2 + 4M_0^2(1 - M_1^2 - G/k)^2]^{\frac{1}{2}}. \tag{4.9}$$

When one approximates the exact solution (3.11) for long waves by noting that $a^2 \cong 1 - 4k$, one obtains (4.9).

In the jet model, the first-order approximation for the wave speed is given by

$$2c^2 = M_\infty^2 + M_{-\infty}^2 + G/k.$$

Marginal stability occurs only when M_∞ and $M_{-\infty}$ are zero and $G/k \rightarrow 0$ as $k \rightarrow 0$. With no background field the approximation to next order gives

$$2c^2 = -k \int_{-\infty}^{\infty} (U^2 - M^2) dz + 2G. \quad (4.10)$$

These results match the results of the heuristic argument given by (3.8) for the long wavelength disturbances of the sinuous wave. Equation (4.10) therefore gives the appropriate long wavelength solution for the sinuous wave of the three-layer jet.

A general form for the approximate solution of the varicose disturbance has not been obtained. Nevertheless, in the case of the three-layer jet it is simple enough to extract from (4.8). Multiplying the results of (4.8) by $[(1-c)^2 - M_0^2]$ produces

$$[(1-c)^2 - M_0^2][c^2 - M_1^2 - G/k] + k[1 - 2c - M_0^2 + M_1^2 + G/k]^2 = 0.$$

One of the roots at $k = 0$ is $c = 1 \pm M_0$. This is marginally stable for $M_0 = 0$ and $G/k \rightarrow 0$. Taking $M_0 = 0$ and noting that since $c - 1 \cong 0$ we can set c equal to zero in all but the first bracket, we then have

$$c = 1 \pm \frac{1}{2}[-4k(1 - M_1^2 - G/k)]^{\frac{1}{2}}.$$

When the exact solution (3.4) is approximated to first order in small k it yields

$$c = 1 - k \pm \frac{1}{2}[-4k(1 - M_1^2 - G/k)]^{\frac{1}{2}};$$

the correspondence between the two formulae for small k is quite satisfying.

Stability boundaries for shear layers

Since the eigenvalue relation (4.8) converges for all k and G such that $c_i \neq 0$ this suggests that we might look more closely at (4.8) for a reasonably accurate estimate of the critical Richardson number. We shall restrict consideration to the case when U and λ are antisymmetric and M is symmetric about the point $z = 0$. The background magnetic field is zero. Physically we can expect instability to be manifest at $c_r = 0$ although for small enough magnetic field values certain examples may exhibit instability as two waves travelling at equal but opposite velocities. Although we shall not prove that $c_r = 0$ when $c_i = 0$ we shall assume it to be true in the profiles we consider. In the limit as $c \rightarrow 0$, equation (4.8) becomes

$$2k - (M_\infty^2 + M_{-\infty}^2) - 2G + \int_{-\infty}^{\infty} [k(U^2 - M^2 - 1 + M_\infty^2) + G(1 - \lambda)] \times [k(U^2 - M^2 - 1 + M_\infty^2) + G(1 + \lambda)] \frac{dz}{U^2 - M^2}. \quad (4.11)$$

Whatever $U = \pm M$ the integrand is singular and thus (4.11) is not strictly speaking integrable. This difficulty can be circumvented by a method used by Drazin & Howard (1961). What is done is to subtract a value from the integrand which is equal to the integrand at the singular point and then, expressing the subtracted quantity as an exact differential, add it outside the integral. Noting first that we can write

$$\frac{1}{U^2 - M^2} = \frac{d}{dz} \left[\ln \frac{U - M}{U + M} \right] \frac{1}{2(MDU - UDM)},$$

and making use of the symmetry properties we have required, (4.11) then becomes

$$G \cong k + k^2 \int_0^\infty \left\{ \left(\frac{X^2 - \lambda^2}{X^2} \right) + \frac{\lambda_{U=M}^2 [MDU - UDM]}{[MDU - UDM]_{U=M} X} \right\} dz. \tag{4.12}$$

We now consider two problems. In the derivative of (4.12) we have assumed differentiability of U, ρ and M , so that we cannot use the discontinuous models. The two problems chosen are among the small number of examples for which the stability boundary is known in the non-magnetic case. Unfortunately, for purposes of comparison no analytic expressions exist for a stability boundary for a magnetic flow.

For Goldstein's problem

$$U = \lambda = \begin{cases} -1, & z \leq -1, \\ z, & |z| < 1, \\ 1, & z > 1. \end{cases}$$

In the limit $M = 0$, equation (4.12) gives $G = k - \frac{2}{3}k^2$ for the stability boundary, which is exact to second order. For Holmboe's problem,

$$U = \lambda = \tanh z.$$

The non-magnetic limit of (4.12) gives $G = k(1 - k)$, which is exact.

Now add to Goldstein's problem a magnetic field given by

$$M = \begin{cases} 0 & |z| > 1, \\ M_0(1 - z), & 0 \leq z \leq 1, \\ M_0(1 + z), & -1 \leq z \leq 0. \end{cases}$$

Equation (4.12) becomes

$$G = k + \frac{1}{3}k^2 - k^2 \left\{ \frac{1}{3}M_0^2 + \frac{1}{1 - M_0^2} + \frac{M_0^2}{(1 - M_0^2)^2} \ln M_0^2 \right\}. \tag{4.13}$$

At $M_0^2 = 0$ this reduces to the non-magnetic result $k - \frac{2}{3}k^2$. For small $M_0^2 > 0$ we see that the magnetic field will have a destabilizing influence because the magnetic field increases the critical stratification. A plot of (4.13) (figure 4) indicates that the magnetic field is destabilizing for all values of $M_0^2 < 1.7$ and stabilizing for $M_0^2 > 1.7$.

Next, consider Holmboe's example with $M = M_0 \operatorname{sech} z$. Equation (4.12) then takes the form

$$G = h + k^2 \int_0^\infty \left[\tanh^2 z - M_0^2 \operatorname{sech}^2 z - \frac{\tanh^2 z}{\tanh^2 z - M_0^2 \operatorname{sech}^2 z} + \frac{M_0^2}{(1 + M_0^2)^{\frac{1}{2}}} \frac{\operatorname{sech} z}{\tanh^2 z - M_0^2 \operatorname{sech}^2 z} \right] dz. \tag{4.14}$$

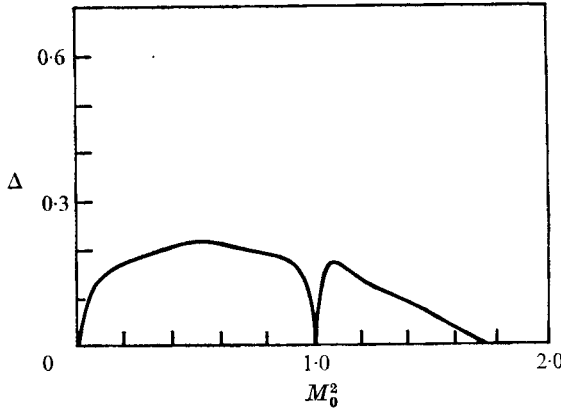


FIGURE 4. Destabilization to $O(k^2)$ for Goldstein's problem.

This has been obtained by noting that at $U = M$ we have

$$\tanh^2 z = M_0^2 \operatorname{sech}^2 z \equiv M_0^2(1 - \tanh^2 z),$$

so that $\tanh^2 z = M_0^2/(1 + M_0^2)$. After all the integrations have been performed, (4.14) reduces to

$$G = k - k^2 \left[1 + M_0^2 - \frac{M_0}{2(1 + M_0^2)^{\frac{1}{2}}} \ln \left(\frac{(1 + M_0^2)^{\frac{1}{2}} + M_0}{(1 + M_0^2)^{\frac{1}{2}} - M_0} \right) \right].$$

Observing that

$$\frac{d}{dM_0} \left\{ M_0 (1 + M_0^2)^{\frac{1}{2}} - \ln \left(\frac{(1 + M_0^2)^{\frac{1}{2}} + M_0}{(1 + M_0^2)^{\frac{1}{2}} - M_0} \right) \right\} = 4 \frac{M_0^2}{(1 + M_0^2)^{\frac{3}{2}}} > 0,$$

for all $M_0^2 > 0$ we find that the critical stratification is smaller with a magnetic field present and thus the magnetic field has a stabilizing effect.

Even for the simple example just considered the computations are quite lengthy. It is possible to obtain some information about the dependence of the stability on the introduction of a small magnetic field without making such lengthy calculations. We take the derivative of G_{crit} with respect to the magnetic field amplitude. When this derivative is positive at zero magnetic field we know that an appropriately small magnetic field is destabilizing.

It is convenient to define $M^2(z) \equiv M_0^2 f^2(z)$ with $f(0) = 1$ and $Df(0) = 0$. Before taking the derivative of (4.12) with respect to M_0^2 we expand the parameters in a Taylor series about $z = 0$.

We obtain

$$\left. \frac{dG}{dM_0^2} \right|_{M_0^2=0} \cong k^2 \int_0^\infty \left\{ f^2(z) \left[-1 - \frac{\lambda^2}{U^4} \right] - \frac{D\lambda(0)}{(DU(0))^2} D \left(\frac{f(z)}{U(z)} \right) \right\} dz. \quad (4.15)$$

Let us return to Holmboe's problem. When $M = M_0 \operatorname{sech} z$, (4.15) gives zero as expected. This gives insufficient information as to the effect which the magnetic field exerts on the fluid and so is useless. However, if $M = M_0 \operatorname{sech}^2 z$, equation (4.15) yields

$$\left. \frac{dG}{dM_0^2} \right|_{M_0^2=0} = \int_0^\infty (2 \operatorname{sech}^2 z - \operatorname{sech}^4 z) dz > 0,$$

and this is greater than zero since the integrand is positive for all z . Thus, a small magnetic field of the form $M_0 \operatorname{sech}^2 z$ will be destabilizing.

5. The solution for continuous problems

When an example does not exhibit instability to normal-mode disturbances it is appropriate to solve the initial-value problem. The method used here is essentially that used by Case (1960), Kent (1968) and others, and thus will merely be outlined here.

In the neighbourhood of a second-order zero of X , equation (2.2) has the solutions

$$F_1 = \eta^{\nu-\frac{1}{2}} \phi_1, \quad F_2 = \eta^{-(\nu+\frac{1}{2})} \phi_2,$$

where $\eta \equiv z - z_0$, $X(z_0) = 0$ and $\phi_{1,2}$ are analytic functions for small η . We define ν by

$$\nu = \left[\frac{1}{4} + \frac{\bar{R}_{i_0}}{(DU_0)^2 - (DM_0)^2} \right]^{\frac{1}{2}}.$$

The subscript zero indicates evaluation of a variable at z_0 . We restrict our attention to the case where $(DU_0)^2 - (DM_0)^2 > 0$ and $M_0 = 0$ so that small changes in c correspond linearly to small changes in η .

The governing equation is given by

$$D[XDF_p] - [k^2X + \bar{R}_i]F_p = (ik)^{-1} \{ [2U + c + ik \partial/\partial t] [D^2 - k^2] F + 2(DU)DF \}_{t=0}, \quad (5.1)$$

where F_p is the Laplace transform of F and the right-hand side of (5.1) represents the initial perturbation. The equation is solved by the method of Green's functions and when the inverse transforms are taken at large time t the dominant behaviour emerges and is given by $F \propto t^{2\nu-1}$, so that for stable stratification the solution decays algebraically in time and for zero stratification $F \propto \text{const}$. The last result differs from that of Kent (1968) only because Kent used a delta-function initial disturbance.

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